



Sensitivity and stability analysis in DEA with interval data

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Abstract

In this paper, we find radius of stability for all decision making units, with interval data. In this approach, organization classification remains unchanged under perturbations of the interval data. Some numerical examples for illustration are given.

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1. Introduction

Data envelopment analysis (DEA), introduced by Charnes et al. [1] (CCR) and extended by Banker et al. [2] (BCC), is a useful method to evaluate relative efficiency of multiple-inputs and multiple-outputs units based on observed data. Its goal is to classify the decision making unit (DMU) into two classes: efficient or inefficient ones. However, uncertainty such as a measurement error should be incorporated in observed data. This indicates the necessity to assess the sensitivity of classifications in DEA. In the recent years, many DEA researchers have studied the sensitivity of efficiency and inefficiency classifications with respect to perturbations in data (see [3–6]). The original DEA

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models assume that inputs and outputs are measured by exact values on a ratio scale.

Recently, Cooper et al. [7] addressed the problem of imprecise data in DEA in its general form. Imprecise data means that some data are known only to the extent that the true values lie within prescribed bounds while other data are known only to satisfy certain ordinal relations. In this paper, we discuss a technique for assessing the sensitivity of efficiency and inefficiency classifications in DEA with interval data.

The focus is on the stability of DMUs that have interval data. Having identified efficient and inefficient DMUs in a DEA analysis with interval data, one may want to know how sensitive these identifications are to possible variations in data.

In this paper, a modified CCR model is suggested to sensitive the DMUs with interval data. We develop several linear programming formulations for investigating radius of stability for all DMUs with interval data. The possible data perturbations for preserving the DMUs classifications are computed from the optimal values.

Interval DEA models are extended to interval data [9]. Then, interval DEA for interval data can be extended to fuzzy data as well, since the level sets of fuzzy data are interval data. Therefore, fuzzy efficiency for fuzzy data can be obtained by interval DEA through the resolution identity [10].

The current article proceeds as follows: In Section 2, we review DEA models for dealing with interval data. Then, on the basis of these models, in Section 3, we propose some models for determining radius of stability for DMUs. In Section 4, the sensitivity and stability analysis methods to several data sets are introduced. A concluding section summarizes our main results.

2. Preliminaries

Suppose we have n DMUs, where each DMU_j ($j \in J$, $J = \{1, \dots, n\}$) consumes m inputs $X_j^T = (x_{1j}, \dots, x_{mj})$ to produce s outputs $Y_j^T = (y_{1j}, \dots, y_{sj})$. We assume $Y_j \geq 0$, $X_j \geq 0$, $Y_j \neq 0$, $X_j \neq 0$. Unlike the original DEA model, we assume further that the levels of inputs and outputs are not known exactly, the true input and output data known to lie within bounded intervals, i.e. $x_{ij} \in [x_{ij}^L, x_{ij}^U]$ and $y_{rj} \in [y_{rj}^L, y_{rj}^U]$ with upper and lower bounds of the intervals given as constants and assumed strictly positive. In this case, the efficiency can be an interval. The upper limit of interval efficiency is obtained from the optimistic viewpoint and the lower limit is obtained from the pessimistic viewpoint. The following model provides such an upper bound for DMU_{j_0} [8]:

$$\begin{aligned}
h_{j_o}^U &= \max \sum_{r=1}^s u_r y_{rj_o}^U \\
\text{s.t.} \quad &\sum_{i=1}^m v_i x_{ij_o}^L = 1, \\
&\sum_{r=1}^s u_r y_{rj_o}^U - \sum_{i=1}^m v_i x_{ij_o}^L \leq 0, \\
&\sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m v_i x_{ij}^U \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
&u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
\end{aligned} \tag{1}$$

where ϵ is a non-Archimedean infinitesimal and original variables $u_1, \dots, u_s$ and $v_1, \dots, v_m$ are weights for outputs and inputs, respectively, which to be estimated. We denote by $h_{j_o}^U$ the efficiency score attained by DMU $_{j_o}$ in (1).

The model below provides a lower bound of the efficiency score for DMU $_{j_o}$:

$$\begin{aligned}
h_{j_o}^L &= \max \sum_{r=1}^s u_r y_{rj_o}^L \\
\text{s.t.} \quad &\sum_{i=1}^m v_i x_{ij_o}^U = 1, \\
&\sum_{r=1}^s u_r y_{rj_o}^L - \sum_{i=1}^m v_i x_{ij_o}^U \leq 0, \\
&\sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
&u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
\end{aligned} \tag{2}$$

The efficiency $h_{j_o}^L$ attained by DMU $_{j_o}$ in (2) serves as a lower bound of its possible efficiency scores. Models (1) and (2) provide each DMU with a bounded interval $[h_{j_o}^L, h_{j_o}^U]$ in which its possible efficiency scores lie from the worst to the best case. Considering (1) and (2), it is evident that $h_{j_o}^L \leq h_{j_o}^U$.

On the basis of the above efficiency score intervals, DMUs can be classified in three subsets as follows:

$$E^{++} = \{j \in J | h_j^L = 1\},$$

$$E^+ = \{j \in J | h_j^L < 1 \quad \text{and} \quad h_j^U = 1\},$$

$$E^- = \{j \in J | h_j^U < 1\}.$$

3. Sensitivity analysis in DEA with interval data

Suppose that DMUs are evaluated by (1) and (2) and are classified in E^{++} , E^+ and E^- . Having identified efficient and inefficient DMUs in a DEA analysis, one may want to know how sensitive these identifications are to possible variations in the data. We determine “radius of stability” within which data variations will not alter a DMU’s classification from efficient to inefficient status or vice versa. We consider the radius of stability of DMU_{j_o} in three cases as follow.

3.1. Radius of stability for DMU belonging to E^{++}

In this case, we assume that DMU_{j_o} is in E^{++} , that is, $h_{j_o}^L = 1$. It is obvious that DMU_{j_o} remains in E^{++} if its outputs increase or its inputs decrease. Our aim is to find the scalars α and β such that if we decrease upper bound of outputs and increase lower bound of inputs of DMU_{j_o} by α then $h_{j_o}^U = 1$, also if we decrease lower bound of outputs and increase upper bound of inputs of DMU_{j_o} by β then $h_{j_o}^L = 1$, i.e. DMU_{j_o} remains in E^{++} .

We will calculate the upper-bounds of α and β , namely the ranges for α and β , such that the inputs and outputs of DMU_{j_o} are perturbed within these ranges and the efficiency of DMU_{j_o} is preserved in E^{++} . It has been assumed that α and β are scalar and strictly positive. Here, we consider the following cases:

- (1) It is obvious if $X_{j_o}^L - \mathbf{1}_m\alpha$, $Y_{j_o}^U + \mathbf{1}_s\alpha$ then $h_{j_o}^U = 1$ and if $X_{j_o}^U - \mathbf{1}_m\beta$, $Y_{j_o}^L + \mathbf{1}_s\beta$ then $h_{j_o}^L = 1$; consequently, $DMU_{j_o} \in E^{++}$ (Note that $\mathbf{1}$ is a vector with all components equal 1.).
- (2) If $X_{j_o}^L + \mathbf{1}_m\alpha$, $Y_{j_o}^U - \mathbf{1}_s\alpha$ and $X_{j_o}^U + \mathbf{1}_m\beta$, $Y_{j_o}^L - \mathbf{1}_s\beta$, then it is possible for DMU_{j_o} not to be in the E^{++} . We are concerned with finding the largest value for α and β such that $DMU_{j_o} \in E^{++}$. For this purpose, the following models are proposed:

$$\begin{aligned}
 & \max \quad \alpha \\
 & \text{s.t.} \quad \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
 & \quad \sum_{r=1}^s u_r (y_{rj_o}^U - \alpha) - \sum_{i=1}^m v_i (x_{ij_o}^L + \alpha) = 0, \\
 & \quad \sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m v_i x_{ij}^U \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
 & \quad u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{3}$$

$$\begin{aligned}
& \max \quad \beta \\
& \text{s.t.} \quad \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
& \quad \sum_{r=1}^s u_r (y_{rj_o}^L - \beta) - \sum_{i=1}^m v_i (x_{ij_o}^U + \beta) = 0, \\
& \quad \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
& \quad u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
\end{aligned} \tag{4}$$

Here, the constraint $\sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1$ is added for normalization purpose. In the first glance, it seems that (3) and (4) are nonlinear, but the following manipulation transforms the models into linear.

$$\begin{aligned}
& \sum_{r=1}^s u_r (y_{rj_o}^U - \alpha) - \sum_{i=1}^m v_i (x_{ij_o}^L + \alpha) = 0, \\
& \sum_{r=1}^s u_r y_{rj_o}^U - \sum_{i=1}^m v_i x_{ij_o}^L - \alpha \left(\sum_{r=1}^s u_r + \sum_{i=1}^m v_i \right) = 0, \\
& \sum_{r=1}^s u_r y_{rj_o}^U - \sum_{i=1}^m v_i x_{ij_o}^L = \alpha.
\end{aligned}$$

Similarly, we have

$$\sum_{r=1}^s u_r y_{rj_o}^L - \sum_{i=1}^m v_i x_{ij_o}^U = \beta.$$

By the above substitutions, the following linear models are obtained:

$$\begin{aligned}
& \max \quad \alpha \\
& \text{s.t.} \quad \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
& \quad \sum_{r=1}^s u_r y_{rj_o}^U - \sum_{i=1}^m v_i x_{ij_o}^L = \alpha, \\
& \quad \sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m v_i x_{ij}^U \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
& \quad u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
\end{aligned} \tag{5}$$

$$\begin{aligned}
& \max \quad \beta \\
& \text{s.t.} \quad \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
& \quad \sum_{r=1}^s u_r y_{rj_o}^L - \sum_{i=1}^m v_i x_{ij_o}^U = \beta, \\
& \quad \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
& \quad u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
\end{aligned} \tag{6}$$

Assuming that α^* and β^* are optimal value of objective functions. Then, DMU $_{j_o}$ with input vector $X_{j_o}^L + \mathbf{1}_m \alpha^*$ and output vector $Y_{j_o}^U - \mathbf{1}_s \alpha^*$ is in E^{++} , similarly DMU $_{j_o}$ with input vector $X_{j_o}^U + \mathbf{1}_m \beta^*$ and output vector $Y_{j_o}^L - \mathbf{1}_s \beta^*$ is in E^{++} .

Theorem 1. For any α , where $\alpha > \alpha^*$, if $x_{ij_o} \in [x_{ij_o}^L + \mathbf{1}_m \alpha, x_{ij_o}^U]$ ($i = 1, \dots, m$) and $y_{rj_o} \in [y_{rj_o}^L, y_{rj_o}^U - \mathbf{1}_s \alpha]$ ($r = 1, \dots, s$) then $h_{j_o}^U < 1$.

Proof. In contrary, suppose there exists α such that $\alpha > \alpha^*$ and $h_{j_o}^U = 1$. Since $h_{j_o}^U = 1$, then there exists v_i, u_r ($i = 1, \dots, m, r = 1, \dots, s$) such that

$$\begin{aligned}
& \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
& \sum_{r=1}^s u_r (y_{rj_o}^U - \alpha) - \sum_{i=1}^m v_i (x_{ij_o}^L + \alpha) = 0, \\
& \sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m v_i x_{ij}^U \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
& u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
\end{aligned}$$

This means that α and v_i, u_r ($i = 1, \dots, m, r = 1, \dots, s$) are feasible solution to (3) and by optimality condition we have $\alpha \leq \alpha^*$, which is a contradiction. $\square$

Theorem 2. For any α , where $\alpha \in [0, \alpha^*]$, if $x_{ij_o} \in [x_{ij_o}^L + \mathbf{1}_m \alpha, x_{ij_o}^U]$ ($i = 1, \dots, m$) and $y_{rj_o} \in [y_{rj_o}^L, y_{rj_o}^U - \mathbf{1}_s \alpha]$ ($r = 1, \dots, s$) then DMU $_{j_o} \in E^{++}$ i.e. $h_{j_o}^U = 1$.

Proof. Suppose that there exists $\alpha \in [0, \alpha^*]$ such that $h_{j_o}^U < 1$. Since $h_{j_o}^U < 1$ then one can find $\lambda \geq 0$ such that

$$\left(\begin{array}{c} -\sum_{j=1}^n \lambda_j X_j^U \\ \sum_{j=1}^n \lambda_j Y_j^L \end{array} \right) \geq \left(\begin{array}{c} -(X_{j_o}^L + \mathbf{1}_m \alpha) \\ Y_{j_o}^U - \mathbf{1}_m \alpha \end{array} \right) \geq \left(\begin{array}{c} -(X_{j_o}^L + \mathbf{1}_m \alpha^*) \\ Y_{j_o}^U - \mathbf{1}_m \alpha^* \end{array} \right),$$

and strict inequality holds at least for one index. Considering model (3), this violates the assumption that DMU_{j_o} efficient one. $\square$

Theorem 3. For any β , where $\beta > \beta^*$, if $x_{ij_o} \in [x_{ij_o}^L, x_{ij_o}^U + \mathbf{1}_m \beta]$ ($i = 1, \dots, m$) and $y_{rj_o} \in [y_{rj_o}^L - \mathbf{1}_s \beta, y_{rj_o}^U]$ ($r = 1, \dots, s$) then $h_{j_o}^L < 1$.

Proof. Considering the similarity between this theorem and Theorem 1 the proof is evident. $\square$

By considering these facts the following theorem holds true as well.

Theorem 4. For any β , where $\beta \in [0, \beta^*]$, if $x_{ij_o} \in [x_{ij_o}^L, x_{ij_o}^U + \mathbf{1}_m \beta]$ ($i = 1, \dots, m$) and $y_{rj_o} \in [y_{rj_o}^L - \mathbf{1}_s \beta, y_{rj_o}^U]$ ($r = 1, \dots, s$) then $DMU_{j_o} \in E^{++}$ i.e. $h_{j_o}^L = 1$.

3.2. Radius of stability for DMU belonging to E^+

In this case, we assume that DMU_{j_o} is in E^+ , that is, $h_{j_o}^L < 1$ and $h_{j_o}^U = 1$. Our aim is to find the scalars α and β such that if we decrease upper bound of outputs and increase lower bound of inputs of DMU_{j_o} by α then $h_{j_o}^U = 1$, also if we increase lower bound of outputs and decrease upper bound of inputs of DMU_{j_o} by β then $h_{j_o}^L < 1$, i.e. DMU_{j_o} remains in E^+ .

We will calculate the upper bounds of α and β , namely the ranges for α and β such that the inputs and outputs of DMU_{j_o} perturbations within these ranges, preserve the efficiency of DMU_{j_o} in E^+ . Here, we consider the following cases:

- (1) If $X_{j_o}^L - \mathbf{1}_m \alpha$, $Y_{j_o}^U + \mathbf{1}_s \alpha$ then $h_{j_o}^U = 1$ and if $X_{j_o}^U$, $Y_{j_o}^L$ are fixed, then $h_{j_o}^L < 1$, consequently $DMU_{j_o} \in E^+$.
- (2) If $X_{j_o}^U + \mathbf{1}_m \beta$, $Y_{j_o}^L - \mathbf{1}_s \beta$ then $h_{j_o}^L < 1$ and if $X_{j_o}^L$, $Y_{j_o}^U$ are fixed then $h_{j_o}^U = 1$, consequently $DMU_{j_o} \in E^+$.
- (3) We are concerned in finding the largest values of α and β such that $DMU_{j_o} \in E^+$.

For this purpose the following models will be considered:

$$\begin{aligned}
 & \max \quad \alpha \\
 & \text{s.t.} \quad \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
 & \quad \sum_{r=1}^s u_r (y_{rj_o}^U - \alpha) - \sum_{i=1}^m v_i (x_{ij_o}^L + \alpha) = 0, \\
 & \quad \sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m v_i x_{ij}^U \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
 & \quad u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{7}$$

$$\begin{aligned}
& \sup \quad \beta \\
& \text{s.t.} \quad \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
& \quad \sum_{r=1}^s u_r (y_{rj_o}^L + \beta) - \sum_{i=1}^m v_i (x_{ij_o}^U - \beta) < 0, \\
& \quad \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
& \quad u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
\end{aligned} \tag{8}$$

It seems that (7) and (8) are nonlinear, with similar manipulation the following linear models are obtained.

$$\begin{aligned}
& \max \quad \alpha \\
& \text{s.t.} \quad \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
& \quad \sum_{r=1}^s u_r y_{rj_o}^U - \sum_{i=1}^m v_i x_{ij_o}^L = \alpha, \\
& \quad \sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m v_i x_{ij}^U \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
& \quad u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
\end{aligned} \tag{9}$$

$$\begin{aligned}
& \sup \quad \beta \\
& \text{s.t.} \quad \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
& \quad \sum_{r=1}^s u_r y_{rj_o}^L - \sum_{i=1}^m v_i x_{ij_o}^U + \beta < 0, \\
& \quad \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
& \quad u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
\end{aligned} \tag{10}$$

Instead of solving (10), the following problem is solved. For optimal value of β , which is called β_1^* , we have $h_j^L = 1$, now we will prove that for all value of $\beta < \beta_1^*$, $h_j^L < 1$.

$$\begin{aligned}
& \min \quad \beta_1 \\
& \text{s.t.} \quad \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
& \quad \sum_{r=1}^s u_r y_{rj_o}^L - \sum_{i=1}^m v_i x_{ij_o}^U + \beta_1 = 0, \\
& \quad \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
& \quad u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
\end{aligned} \tag{11}$$

Assume α^* and β_1^* are optimal value of objective functions. Then DMU_{j_o} with input vector $X_{j_o}^L + \mathbf{1}_m \alpha^*$ and output vector $Y_{j_o}^U - \mathbf{1}_s \alpha^*$ is in E^+ , similarly DMU_{j_o} with input vector $X_{j_o}^U - \mathbf{1}_m \beta$ ($\beta < \beta_1^*$) and output vector $Y_{j_o}^L + \mathbf{1}_s \beta$ ($\beta < \beta_1^*$) is in E^+ . The following theorems assert some concept concerning the fact stated above and some other result which are needed for our purpose.

Theorem 5. For any α , where $\alpha > \alpha^*$, if $x_{ij_o} \in [x_{ij_o}^L + \mathbf{1}_m \alpha, x_{ij_o}^U]$ and $y_{rj_o} \in [y_{rj_o}^L, y_{rj_o}^U - \mathbf{1}_s \alpha]$, then $h_{j_o}^U < 1$.

Proof. Proof is evident. $\square$

Theorem 6. For any α , where $\alpha \in [0, \alpha^*]$, if $x_{ij_o} \in [x_{ij_o}^L + \mathbf{1}_m \alpha, x_{ij_o}^U]$ and $y_{rj_o} \in [y_{rj_o}^L, y_{rj_o}^U - \mathbf{1}_s \alpha]$, then $\text{DMU}_{j_o} \in E^+$ i.e. $h_{j_o}^U = 1$.

Proof. Proof is evident. $\square$

Theorem 7. For any β , where $\beta < \beta_1^*$, if $x_{ij_o} \in [x_{ij_o}^L, x_{ij_o}^U - \mathbf{1}_m \beta]$ and $y_{rj_o} \in [y_{rj_o}^L + \mathbf{1}_s \beta, y_{rj_o}^U]$, then $\text{DMU}_{j_o} \in E^+$, i.e. $h_{j_o}^L < 1$.

Proof. Suppose there exists β such that $\beta < \beta_1^*$ and $h_{j_o}^L = 1$. Since $h_{j_o}^L = 1$, then there exists v_i, u_r ($i = 1, \dots, m, r = 1, \dots, s$) such that

$$\begin{aligned}
& \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
& \sum_{r=1}^s u_r (y_{rj_o}^L + \beta) - \sum_{i=1}^m v_i (x_{ij_o}^U - \beta) = 0, \\
& \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
& u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
\end{aligned}$$

This means that β and v_i, u_r ($i = 1, \dots, m, r = 1, \dots, s$) are feasible solutions to (11) and by optimality condition we have $\beta \geq \beta_1^*$ which is a contradiction. $\square$

3.3. Radius of stability of DMU belonging to E^-

In this case, we assume that DMU_{j_o} is in E^- . Our aim is to find the scalars α and β such that if we decrease lower bound of inputs and increase upper bound of outputs of DMU_{j_o} by α then $h_{j_o}^U < 1$, also if we decrease upper bound of inputs and increase lower bound of outputs of DMU_{j_o} by β then $h_{j_o}^L < 1$, i.e. DMU_{j_o} remains in E^- .

We will calculate the upper bounds of α and β , namely the ranges for α and β such that the inputs and outputs of DMU_{j_o} are perturbed within these ranges and its inefficiency score is not changed. Now we consider following cases:

- (1) If $X_{j_o}^L + \mathbf{1}_m \alpha, Y_{j_o}^U - \mathbf{1}_s \alpha$ then $h_{j_o}^U < 1$ and if $X_{j_o}^U + \mathbf{1}_m \beta, Y_{j_o}^L - \mathbf{1}_s \beta$ then and $h_{j_o}^L < 1$, consequently $DMU_{j_o} \in E^-$.
- (2) We are concerned with finding the largest value for α and β such that $DMU_{j_o} \in E^-$.

We consider the following models:

$$\begin{aligned}
 & \sup \quad \alpha \\
 & \text{s.t.} \quad \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
 & \quad \sum_{r=1}^s u_r (y_{rj_o}^U + \alpha) - \sum_{i=1}^m v_i (x_{ij_o}^L - \alpha) < 0, \\
 & \quad \sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m v_i x_{ij}^U \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
 & \quad u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{12}$$

$$\begin{aligned}
 & \sup \quad \beta \\
 & \text{s.t.} \quad \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
 & \quad \sum_{r=1}^s u_r (y_{rj_o}^L + \beta) - \sum_{i=1}^m v_i (x_{ij_o}^U - \beta) < 0, \\
 & \quad \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
 & \quad u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{13}$$

Similarly, the above mentioned model can be transformed to following linear models:

$$\begin{aligned}
 & \sup \quad \alpha \\
 & \text{s.t.} \quad \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
 & \quad \sum_{r=1}^s u_r y_{rj_o}^U - \sum_{i=1}^m v_i x_{ij_o}^L + \alpha < 0, \\
 & \quad \sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m v_i x_{ij}^U \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
 & \quad u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{14}$$

$$\begin{aligned}
 & \sup \quad \beta \\
 & \text{s.t.} \quad \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
 & \quad \sum_{r=1}^s u_r y_{rj_o}^L - \sum_{i=1}^m v_i x_{ij_o}^U + \beta < 0, \\
 & \quad \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
 & \quad u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{15}$$

Since the second constraints in (14) and (15) are strict inequality, then these models are nonlinear programming and the following linear programming problems are suggested:

$$\begin{aligned}
 & \min \quad \alpha_1 \\
 & \text{s.t.} \quad \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
 & \quad \sum_{r=1}^s u_r y_{rj_o}^U - \sum_{i=1}^m v_i x_{ij_o}^L + \alpha_1 = 0, \\
 & \quad \sum_{r=1}^s u_r y_{rj}^L - \sum_{i=1}^m v_i x_{ij}^U \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
 & \quad u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
 \end{aligned} \tag{16}$$

$$\begin{aligned}
& \min \quad \beta_1 \\
& \text{s.t.} \quad \sum_{r=1}^s u_r + \sum_{i=1}^m v_i = 1, \\
& \quad \sum_{r=1}^s u_r y_{rj_o}^L - \sum_{i=1}^m v_i x_{ij_o}^U + \beta_1 = 0, \\
& \quad \sum_{r=1}^s u_r y_{rj}^U - \sum_{i=1}^m v_i x_{ij}^L \leq 0, \quad j = 1, \dots, n, \quad j \neq j_o, \\
& \quad u_r, v_i \geq \epsilon, \quad r = 1, \dots, s, \quad i = 1, \dots, m.
\end{aligned} \tag{17}$$

Assume α_1^* and β_1^* are optimal value of objective functions. Then DMU_{j_o} with input vector $X_{j_o}^L - \mathbf{1}_m \alpha$ ($\alpha < \alpha_1^*$) and output vector $Y_{j_o}^U + \mathbf{1}_s \alpha$ ($\alpha < \alpha_1^*$) is in E^- , similarly DMU_{j_o} with input vector $X_{j_o}^U - \mathbf{1}_m \beta$ ($\beta < \beta_1^*$) and output vector $Y_{j_o}^L + \mathbf{1}_s \beta$ ($\beta < \beta_1^*$) is in E^- .

Theorem 8. For any α , where $\alpha < \alpha_1^*$, if $x_{ij_o} \in [x_{ij_o}^L - \mathbf{1}_m \alpha, x_{ij_o}^U]$ and $y_{rj_o} \in [y_{rj_o}^L, y_{rj_o}^U + \mathbf{1}_s \alpha]$, then $\text{DMU}_{j_o} \in E^-$ i.e. $h_{j_o}^U < 1$, where α_1^* is the optimal value to (16).

Proof. Considering the similarity between this theorem and Theorem 7 the proof is omitted. $\square$

Theorem 9. For any β , where $\beta < \beta_1^*$, if $x_{ij_o} \in [x_{ij_o}^L, x_{ij_o}^U - \mathbf{1}_m \beta]$ ($i = 1, \dots, m$) and $y_{rj_o} \in [y_{rj_o}^L + \mathbf{1}_s \beta, y_{rj_o}^U]$ ($r = 1, \dots, s$) then $\text{DMU}_{j_o} \in E^-$ i.e. $h_{j_o}^L < 1$, where β_1^* is the optimal value to (17).

Proof. Proof is evident. $\square$

Table 1
Data of numerical example

DMU _j	Input				Output				Efficiency unit		Classification
	x_{1j}^L	x_{1j}^U	x_{2j}^L	x_{2j}^U	y_{1j}^L	y_{1j}^U	y_{2j}^L	y_{2j}^U	h_j^L	h_j^U	
1	12	15	0.21	0.48	138	144	21	22	0.224	1	E^+
2	10	17	0.1	0.7	143	159	28	35	0.227	1	E^+
3	4	12	0.16	0.35	157	198	21	29	0.823	1	E^+
4	19	22	0.12	0.19	158	181	21	25	0.445	0.907	E^-
5	14	15	0.06	0.09	157	161	28	40	1	1	E^{++}

Table 2
The sensitivity analysis result for DMUs

DMU _j	Input				Output				Efficiency	
	x_{1j}^L	x_{1j}^U	x_{2j}^L	x_{2j}^U	y_{1j}^L	y_{1j}^U	y_{2j}^L	y_{2j}^U	h_j^L	h_j^U
5	19.63626	15.00872	5.69626	0.09872	156.99128	155.36374	27.99128	34.36374	1	1

Table 3
The values of α^* and β_{ij}^* for DMU_j

DMUs	α^*	β_1^*
1	0.12445	0.42843
2	5.69766	0.64648
3	13.61625	0.27068

Table 4
The sensitivity analysis results for DMU_j

DMU _j	Input				Output				Efficiency	
	x_{1j}^L	x_{1j}^U	x_{2j}^L	x_{2j}^U	y_{1j}^L	y_{1j}^U	y_{2j}^L	y_{2j}^U	h_j^L	h_j^U
1	12.12445	14.5719	0.33445	0.0519	138.4281	143.87555	21.4281	21.87555	0.994	1
2	15.69766	16.355	5.79766	0.055	143.645	153.30234	28.645	29.30234	0.973	1
3	17.61625	11.74	13.77625	0.09	157.26	184.38375	21.26	15.38375	0.950	1

Table 5
The sensitivity analysis result for DMU₄

DMU _j	Input				Output				Efficiency	
	x_{1j}^L	x_{1j}^U	x_{2j}^L	x_{2j}^U	y_{1j}^L	y_{1j}^U	y_{2j}^L	y_{2j}^U	h_j^L	h_j^U
4	18.9839	21.87	0.1039	0.06	158.13	181.0161	21.13	25.0161	0.982	0.999

4. Numerical example

Consider the interval data setting of Table 1 contain 5 DMUs with 2 inputs and 2 outputs and the efficiency scores obtained by applying (1) and (2) [8].

Considering the data from Table 1 and using models in Section 3, we calculate the radius of stability for all DMUs as the following:

(i) *Radius of stability E^{++}*

For determining radius of stability of DMU₅, we apply (5) and (6) (because DMU₅ is in E^{++}). In this case, $\alpha^* = 5.63626$ and $\beta^* = 0.00872$. If x_{i5}^L , for $i = 1, 2$ are increased by α^* and y_{r5}^U , for $r = 1, 2$ are decreased by α^* , and apply (1), then $h_5^U = 1$ and DMU₅ remains in E^{++} . If x_{i5}^U , for $i = 1, 2$ are increased by β^* , and y_{r5}^L , for $r = 1, 2$ are decreased by β^* , and apply (2), then $h_5^L = 1$, consequently DMU₅ remains in E^{++} .

Table 2, reports the sensitivity analysis result for DMU₅.

(ii) *Radius of stability E^+*

By applying (9) and (11) (because DMU's 1, 2 and 3 are in E^+) the data is as shown in Table 3.

If x_{ij}^L , for $i = 1, 2$ and $j = 1, 2, 3$ are increased by α_j^* and y_{rj}^U , for $r = 1, 2$ and $j = 1, 2, 3$ are decreased by α_j^* , and apply (1), then $h_j^U = 1$ and DMU_j remains in E^+ . If x_{ij}^U , for $i = 1, 2$ and $j = 1, 2, 3$ are decreased by β_j ($\beta_j < \beta_j^*$) and y_{rj}^L , for $r = 1, 2$ and $j = 1, 2, 3$ are increased by β_j ($\beta_j < \beta_j^*$), and apply (2), then $h_j^L < 1$, consequently DMU_j remains in E^+ .

Table 4 reports the sensitivity analysis result for the DMU_j where $j = 1, 2, 3$. In this case we assume $\beta_1 = 0.4281$, $\beta_2 = 0.645$ and $\beta_3 = 0.26$.

(iii) *Radius of stability E^-*

For determining radius of stability of DMU₄, we apply (16) and (17) (because DMU₄ is in E^-). In this case, $\alpha_1^* = 0.01624$ and $\beta_1^* = 0.1311$. If x_{i4}^L , for $i = 1, 2$ are decreased by α ($\alpha < \alpha_1^*$) and y_{r4}^U , for $r = 1, 2$ are increased by α ($\alpha < \alpha_1^*$) and apply (1) then $h_4^U < 1$ and DMU₄ remains in E^- . If x_{i4}^U , for $i = 1, 2$ are decreased by β ($\beta < \beta_1^*$) and y_{r4}^L , for $r = 1, 2$ are increased by β ($\beta < \beta_1^*$), and apply (2) then $h_4^L < 1$ consequently DMU₄ remains in E^- .

Table 5 reports the sensitivity analysis result for the DMU₄. In this case, we assume $\alpha = 0.0161$ and $\beta = 0.13$.

5. Conclusion

In this paper, we proposed a method for finding the largest region for preserving classification of DMU with interval data. For this purpose, the classical CCR model is modified. Finding largest region which preserves the

classification, for each DMU a radius of stability is found. Such a work with fuzzy data may be done and the model may be extended.

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